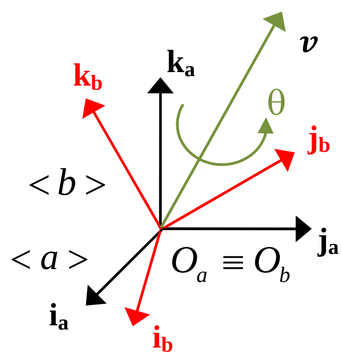
Versor Lemma

Given two frames $< a >$ and $< b >$, with the latter rotated with respect to the former of an angle θ around the versor $\boldsymbol{v}$**,** the following relations hold:



$$\begin{cases} (\boldsymbol{i}_a \wedge \boldsymbol{i}_b) + (\boldsymbol{j}_a \wedge \boldsymbol{j}_b) + (\boldsymbol{k}_a \wedge \boldsymbol{k}_b) = 2\boldsymbol{v} sin(\theta) & (1) \\ (\boldsymbol{i}_a \cdot \boldsymbol{i}_b) + (\boldsymbol{j}_a \cdot \boldsymbol{j}_b) + (\boldsymbol{k}_a \cdot \boldsymbol{k}_b) = 1 + 2cos(\theta) & (2) \end{cases}$$

Algorithm

It is possible to compute $\boldsymbol{v}$ and $\theta \in (-\pi, \pi)$ in the following way.

1) Compute first the left-hand side of (2) and solve (2) for $\delta \triangleq \cos{(\theta)}$.
2) If $\delta = 1$ then we can assume:

   $$(\boldsymbol{v}, \theta) = (\forall, 0) \quad (3)$$
3) If $|\delta| < 1$ compute the left-hand side of (1) and solve the (1) for $\boldsymbol{\sigma} \triangleq 2\, \boldsymbol{v} \sin(\theta)$ and our solution is:

   $$(\boldsymbol{v}, \theta) = \pm \left[\frac{\boldsymbol{\sigma}}{|\boldsymbol{\sigma}|};\ \tan^{-1}\left(\frac{|\boldsymbol{\sigma}|}{2}, \delta \right) \right] \quad (4)$$
4) If $\delta = -1$ we have

   $$(\boldsymbol{v}, \theta) = \pm(\boldsymbol{v}_0; \pm\pi) \quad (5)$$

   where:

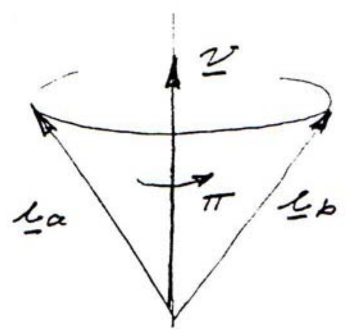
   $$\boldsymbol{v}_0 \triangleq \frac{(\boldsymbol{i}_a + \boldsymbol{i}_b) + (\boldsymbol{j}_a + \boldsymbol{j}_b) + (\boldsymbol{k}_a + \boldsymbol{k}_b)}{|(\boldsymbol{i}_a + \boldsymbol{i}_b) + (\boldsymbol{j}_a + \boldsymbol{j}_b) + (\boldsymbol{k}_a + \boldsymbol{k}_b)|} \quad (6)$$

Proof of (6).

The algorithm is easy to understand, except for the (6) that needs few more considerations.

In case $\delta = -1$, obviously the rotation angle can be only $\theta = \pm\pi$, and the vector $\boldsymbol{\sigma} \triangleq 2\, \boldsymbol{v} \sin(\theta)$ is zero. Thus it is impossible to use the (4).

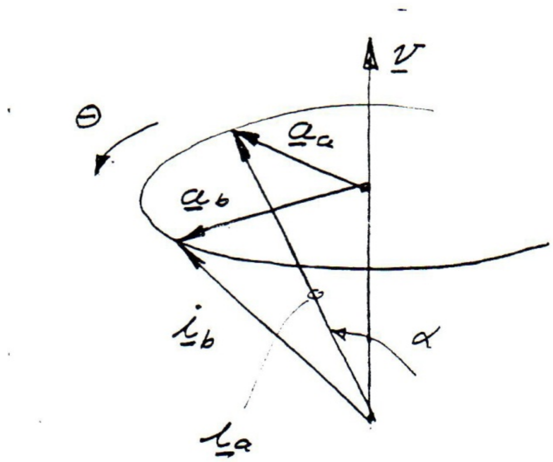
However, we can note that, when $\theta = \pm\pi$, every pair of corresponding frame axis of the two frames, is necessarily placed overa diameter of a circle bisected by the axis $\boldsymbol{v}$:



The (5) follows immediately for geometric construction.

In the (6) we chosen to add and normalize all the vectors. This is more efficient of using the sum of only a pair of axis, which can be zero in case the axis of rotation $\boldsymbol{v}$ coincides with one of the axis of the frame.

Proof of (1)



Consider the revolution of the versor $\boldsymbol{i}_b$ for moving from its initial position (coincident with $\boldsymbol{i}_a$) to the final. We have:

$$\boldsymbol{a}_a \wedge \boldsymbol{a}_b = \boldsymbol{v}\ \sin^2\alpha\ \sin\theta$$

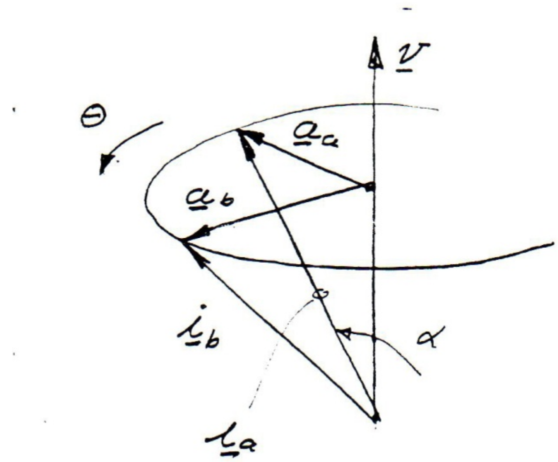
Similarly:

$$\boldsymbol{b}_a \wedge \boldsymbol{b}_b = \boldsymbol{v}\ \sin^2\beta\ \sin\theta$$
$$\boldsymbol{c}_a \wedge \boldsymbol{c}_b = \boldsymbol{v}\ \sin^2\gamma\ \sin\theta$$

Adding the 3 equations:

$$\boldsymbol{a}_a \wedge \boldsymbol{a}_b + \boldsymbol{b}_a \wedge \boldsymbol{b}_b + \boldsymbol{c}_a \wedge \boldsymbol{c}_b =$$
$$= \boldsymbol{v}\sin\theta\,(\sin^2\alpha + \sin^2\beta + \sin^2\gamma) =$$
$$= \ \boldsymbol{v}\sin\theta\,(3 - \cos^2\alpha + \cos^2\beta + \cos^2\gamma) =$$
$$= \ \boldsymbol{2v}\sin\theta$$

since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, because it is the modulus of $\boldsymbol{v}$. In fact, the quantity $|\boldsymbol{v}|\cos\alpha = \cos\alpha$ is the projection of $\boldsymbol{v}$ onto $\boldsymbol{i}_a$, and so on.



Note that:

$$\begin{aligned}
&\boldsymbol{a}_a \wedge \boldsymbol{a}_b = \\
&= [\boldsymbol{i}_a - (\boldsymbol{i}_a \cdot \boldsymbol{v})\boldsymbol{v}] \wedge [\boldsymbol{i}_b - (\boldsymbol{i}_b \cdot \boldsymbol{v})\boldsymbol{v}] = \\
&= \boldsymbol{i}_a \wedge \boldsymbol{i}_b - \boldsymbol{i}_b \wedge (\boldsymbol{i}_a \cdot \boldsymbol{v})\boldsymbol{v} - \boldsymbol{i}_a \wedge (\boldsymbol{i}_b \cdot \boldsymbol{v})\boldsymbol{v} = \\
&= \boldsymbol{i}_a \wedge \boldsymbol{i}_b - (\boldsymbol{i}_a - \boldsymbol{i}_b) \wedge (\boldsymbol{i}_a \cdot \boldsymbol{v})\boldsymbol{v}
\end{aligned}$$

where the last follows from $(\boldsymbol{i}_a \cdot \boldsymbol{v}) = (\boldsymbol{i}_b \cdot \boldsymbol{v})$, since $\boldsymbol{v}$ has the same components on either $< a >$ and $< b >$.

Similarly:

$$\boldsymbol{b}_a \wedge \boldsymbol{b}_b = \boldsymbol{j}_a \wedge \boldsymbol{j}_b - (\boldsymbol{j}_a - \boldsymbol{j}_b) \wedge (\boldsymbol{j}_a \cdot \boldsymbol{v})\boldsymbol{v}$$

$$\boldsymbol{c}_a \wedge \boldsymbol{c}_b = \boldsymbol{k}_a \wedge \boldsymbol{k}_b - (\boldsymbol{k}_a - \boldsymbol{k}_b) \wedge (\boldsymbol{k}_a \cdot \boldsymbol{v})\boldsymbol{v}$$

Adding:

$$\begin{aligned}
&\boldsymbol{a}_a \wedge \boldsymbol{a}_b + \boldsymbol{b}_a \wedge \boldsymbol{b}_b + \boldsymbol{c}_a \wedge \boldsymbol{c}_b = \\
&= \boldsymbol{i}_a \wedge \boldsymbol{i}_b + \boldsymbol{j}_a \wedge \boldsymbol{j}_b + \boldsymbol{k}_a \wedge \boldsymbol{k}_b + \\
&-[(\boldsymbol{i}_a \cdot \boldsymbol{v})(\boldsymbol{i}_a - \boldsymbol{i}_b) + (\boldsymbol{j}_a \cdot \boldsymbol{v})(\boldsymbol{j}_a - \boldsymbol{j}_b) + (\boldsymbol{k}_a \cdot \boldsymbol{v})(\boldsymbol{k}_a - \boldsymbol{k}_b)] \wedge \boldsymbol{v}
\end{aligned}$$

Note that the term in the bracket is $\boldsymbol{v} - \boldsymbol{v} = 0$: so the first part of the lemma (1) follows.

The proof of the second equation (2) of the lemma is similar.

(left as exercise).